Relation of Z-transform and Laplace Transform in Discrete Time Signal with property R.O.C

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Abstract – An introduction to Z and Laplace transform, there relation with properties of ROC is the topic of this paper. It deals with a review of what z-transform plays role in the analysis of discrete-time single and LTI system as the Laplace transform does in the analysis of continuous-time signals and L.T.I. And what does the specific region of convergence represent. A pictorial representation of the region of convergence has been sketched and relation is discussed.

Index Terms – Z-Transform, Laplace Transform, discrete-time signal, Region of convergence.

1. INTRODUCTION

Z-transform, like the Laplace transform, is an indispensable mathematical tool for the design, analysis and monitoring of systems. The z-transform is the discrete-time counter-part of the Laplace transform and a generalisation of the Fourier transform of a sampled signal. Like Laplace transform the z-transform allows insight into the transient behaviour, the steady state behaviour, and the stability of discrete-time systems. A working knowledge of the z-transform is essential to the study of digital filters and systems. This paper begins with the definition of the derivation of the z-transform from the Laplace transform of a discrete-time signal. A useful aspect of the Laplace and the z-transforms are there presentation of a system in terms of the locations of the poles and the zeros of the system transfer function in a complex plane.[1,6].

2. DERIVATION OF THE Z-TRANSFORM

The z-transform is the discrete-time counterpart of the Laplace transform. In this section we derive the z-transform from the Laplace transform a discrete-time signal. The Laplace transform $X(s)$, of a continuous-time signal $x(t)$, is given by the integral.

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt \qquad (1)$$

where the complex variable $s=\sigma + j\omega$, and the lower limit of $t=0^-$ allows the possibility that the signal $x(t)$ may include an impulse.[1,5,6]

The inverse Laplace transform is defined by

$$x(t) = \int_{\sigma_{1-\infty}}^{\sigma_{1+\infty}} X(s)e^{-st}ds \qquad (2)$$

where σ_1 is selected so that $X(s)$ is analytic (no singularities) for $s>\sigma_1$. The z-transform can be derived from Eq. (1) by sampling the continuous-time input signal $x(t)$. For a sampled signal $x(mT_s)$, normally denoted as $x(m)$ assuming the sampling period $T_s=1$, the Laplace transform Eq. (1) becomes

$$X[e^s] \equiv \sum_{m=0}^{\infty} x[m]e^{-sm} \qquad (3)$$

Substituting the variable e^s in Eq. (3) with the variable z we obtain the one-sided z-transform equation

$$X[z] \equiv \sum_{m=0}^{\infty} x[m]z^{-m} \qquad (4)$$

The two-sided z-transform is defined as

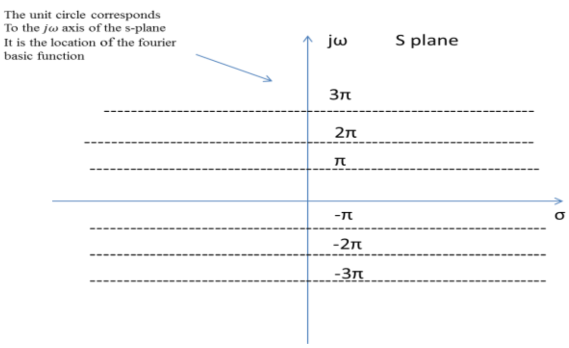
$$X[z] \equiv \sum_{m=-\infty}^{\infty} x[m]z^{-m} \qquad (5)$$

Note that for a one-sided signal, $x(m)=0$ for $m<0$, Eqs. (4) and (5) are equivalent.

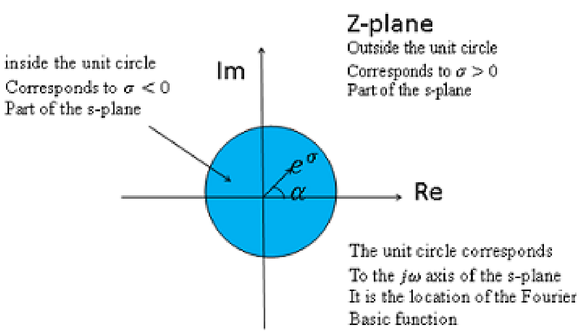
A similar relationship exists between the Laplace transform and the Fourier transform of a continuous time signal. The Laplace transform is a one-sided transform with the lower limit of integration at $t = 0^-$, whereas the Fourier transform (1,2) is a two-sided transform with the lower limit of integration at $t = -\infty$. However for a one-sided signal, which is zero-valued for $t< 0^-$, the limits of integration for the Laplace and the Fourier transforms are identical. In that case if the variable s in the Laplace transform is replaced with the frequency variable $j2\pi f$ then the Laplace integral becomes the Fourier integral. Hence for a one-sided signal, the Fourier transform is a special case of the Laplace transform corresponding to $s=j2\pi f$ and $\sigma=0$.[1,5,6]

3. THE Z-PLANE AND THE UNIT CIRCLE

The frequency variables of the Laplace transform $s=\sigma + j\omega$, and the z-transform $z=re^{j\omega}$ are complex variables with real and imaginary parts and can be visualised in a two dimensional plane. The *s*-plane of the Laplace transform and the *z*-plane of z-transform. In the *s*-plane the vertical $j\omega$−axis is the frequency axis, and the horizontal σ-axis gives the exponential rate of decay, or the rate of growth, of the amplitude of the complex sinusoid as also shown in Fig. 1. As shown



Fig(1)



Fig(2)

Figure - Illustration of (1) the S-plane and (2) the Z-plane.

when a signal is sampled in the time domain its Laplace transform, and hence the *s*-plane, becomes periodic with respect to the $j\omega$−axis. This is illustrated by the periodic horizontal dashed lines in Fig 1. Periodic processes can be conveniently represented using a circular polar diagram such as the z-plane and its associated unit circle. Now imagine bending the $j\omega$−axis of the *s*-plane of the sampled signal of Fig. 1 in the direction of the left hand side half of the *s*-plane to form a circle such that the points π and $-\pi$ meet. The resulting circle is called the *unit circle*, and the resulting diagram is called the *z*-plane. The area to the left of the *s*-plane, i.e. for $\sigma< 0$ or $r=e^{\sigma}<1$, is mapped into the area inside the unit circle, this is the region of stable causal signals and systems. The area to the right of the *s*-plane, $\sigma> 0$ or $r=e^{\sigma}>1$, is mapped onto the outside of the unit circle this is the region of unstable signals and systems. The $j\omega$−axis, with $\sigma=0$ or $r=e^{\sigma}=1$, is itself mapped onto the unit circle line. Hence the Cartesian co-ordinates used in s-plane for continuous time signals Fig. 1, is mapped into a polar representation in the z-plane for discrete-time signals Fig.2. [1]

4. THE REGION OF CONVERGENCE (ROC)

Since the z-transform is an infinite power series, it exists only for those values of the variable *z* for which the series converges to a finite sum. The region of convergence (ROC) of $X(z)$ is the set of all the values of *z* for which $X(z)$ attains a finite computable value.[1,2,3]

To find the value of z for which the series converges, we use the ratio test or the root test states that a series of complex number

$$\sum_{m=0}^{\infty} a_m$$

With limit $$\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = A \qquad (6)$$

Converges absolutely if A<1 and diverges if A>1 the series may or may not converge. The root test state that if $\lim_{m\to\infty}\sqrt[m]{|a_m|} = A$ (7)

Then the series converges absolutely if A<1, and diverges if A>1, and may converge or diverge if A=1.

More generally, the series converges absolutely if

$$\overline{\lim_{m\to\infty}}\sqrt[m]{|a_m|} < 1 \qquad (8)$$

Where $\overline{lim}$ denotes the greatest limit points of $\overline{\lim_{m\to\infty}}|x(mT)|^{1/m}$,

and diverges if $$\overline{\lim_{m\to\infty}}\sqrt[m]{|a_m|} > 1 \qquad (9)$$

If we apply the root test in equation (4) we obtain the convergence condition

$$\overline{\lim_{m\to\infty}}\sqrt[m]{|x(mT)z^{-m}|} = \overline{\lim_{m\to\infty}}\sqrt[m]{|x(mT)||z^{-1}|^m} < 1$$

$$|z| > \overline{\lim_{m\to\infty}}\sqrt[m]{|x(mT)|} = R \qquad (10)$$

Where R is known as the radius of convergence for the series. Therefore the series will converge absolutely for all points in the z-plane that lie outside the circle of radius R, and is centred at the origin(with the possible exception of the point at infinity). This region is called the region of convergence (ROC).

5. PROPERTIES OF ROC OF Z-TRANSFORMS

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.

- If x(m) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.
- If x(m) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.
- If x(m) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(m) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x(m) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

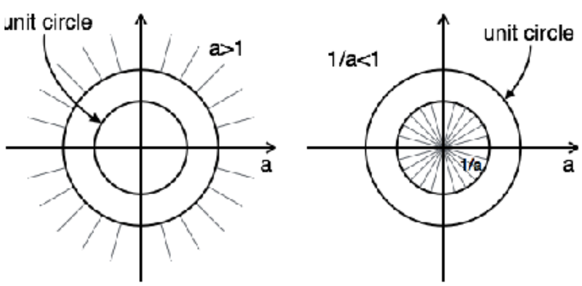
The concept of ROC can be explained by the following example:

Example 1: Find z-transform and ROC of $anu[n]+a-nu[-n-1]$

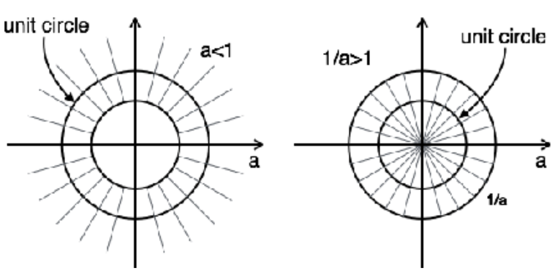
$Z.T[anu[n]]+Z.T[a-nu[-n-1]]=ZZ-a+ZZ-1a$

$ROC:|z|>aROC:|z|<1a$

The plot of ROC has two conditions as a > 1 and a < 1, as you do not know a.



In this case, there is no combination ROC.



Here, the combination of ROC is from $a<|z|<1a$

Hence for this problem, z-transform is possible when a < 1.

6. CAUSALITY AND STABILITY

Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when

- ROC is outside the outermost pole.
- In The transfer function H[Z], the order of numerator cannot be greater than the order of denominator.

7. STABILITY CONDITION FOR DISCRETE TIME LTI SYSTEMS

A discrete time LTI system is stable when

- Its system function H[Z] include unit circle |z|=1.
- All poles of the transfer function lay inside the unit circle |z|=1.

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